MATH 122B: HOMEWORK 2

Suggested due date: August 15th, 2016

(1) Compute the integral
$$\int_C \frac{e^{2z} + \sin(z)}{z - \pi} dz,$$
where C is the circle $|z - 2| = 2$ traversed once in the counter clockwise direction.

(2) Compute
$$\int_C \frac{e^z}{z(z-2)} dz$$
where C is a simple closed curve containing the point 0 and 2.

(3) Let D be an open connected set and C a closed curve in D. Suppose f is holomorphic on D and the derivative $f'(z)$ is continuous on D. Show that
$$I = \int_C \overline{f(z)} f'(z) dz$$
is purely imaginary.

(4) Evaluate
$$\int_C \frac{e^z + e^{-z}}{2z^4} dz$$
where C is any simple closed curve enclosing 0.

(5) Let f be holomorphic inside and on the unit circle C. Show that, for $0 < |z| < 1$,
$$2\pi i f(z) = \int_C \frac{f(w)}{w - z} dw - \int_C \frac{f(w)}{(w - \frac{1}{\bar{z}}} dw.$$
From this, deduce the **Poisson integral formula**
$$f(re^{i\theta}) = \frac{1}{2\pi} \int_0^{2\pi} \frac{(1 - r^2)}{1 - 2r\cos(\theta - t) + r^2} f(e^{it}) dt, \qquad \text{for } 0 < r < 1.$$

(6) If $f(z)$ is holomorphic and $|f(z)| \leq \dfrac{1}{1 - |z|}$ in $|z| < 1$, show that $|f'(0)| \leq 4$.

(7) Let $f(z)$ be holomorphic on $\mathbb{C}$ such that $|f'(z)| \leq |z|$. Show that $f(z) = a + bz^2$ with some constants $a, b \in \mathbb{C}$ such that $|b| \leq 1$.

(8) Find the Laurent series of $\dfrac{e^{2z}}{(z-1)^3}$ around $z = 1$.

(9) Evaluate
$$\int_C \frac{z^2}{(z-1)^2(z+2)} dz$$
where C is the curve $|z| = 3$, oriented positively.

Solutions

(1) Since $\pi < 4$, by Cauchy integral formula, $2\pi i e^{2\pi}$.

(2) By residue theorem, $2\pi i(\frac{e^2}{2} - \frac{1}{2})$.

(3)

$$\int_C \overline{f(z)} df(z) = -\int_C d(\overline{f}) f(z) = -\int_C f(z) \overline{f'(z)} d\bar{z}$$

hence is equal to negative of its conjugate (why are the equations valid?). A more elementary method was done in class.

(4) By series expansion,

$$\begin{aligned} e^z &= 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \cdots \\ e^{-z} &= 1 - z + \frac{z^2}{2} - \frac{z^3}{6} + \cdots \\ e^z + e^{-z} &= 2 + z^2 + O(z^4) \end{aligned}$$

hence the $1/z$ component of $\dfrac{e^z + e^{-z}}{2z^4}$ is 0, hence the answer is 0. There are other ways to do this as well, for instance, this is the 3rd derivative of $\cosh(z)$ at $z = 0$, which is $\sinh(0) = 0$.

(5) Since $|1/\bar{z}| > 1$, it is outside the unit disk so the second integral is 0.

(6) Use Cauchy integral formula for the derivative for a circle of radius $\frac{1}{2}$ centered at 0.

(7) By Cauchy integral formula for the derivative,

$$\begin{aligned} |f''(z)| &= \left| \frac{1}{2\pi i} \int_{|w-z|=R} \frac{f'(w)}{(w-z)^2} dw \right| \\ &\leq \frac{1}{2\pi R^2}(R + |z|) \int_{|w-z|=R} dw \leq 1 + \frac{|w|}{R} \leq 2 \end{aligned}$$

For R sufficiently large. Hence f'' is bounded and entire, hence constant. Use $|f'(0)| \leq 0$ to show that the linear term is zero.

(8) One way to compute is

$$e^{2z} = e^{2(z-1)+2} = e^2 \sum_{n=0}^{\infty} \frac{(2(z-1))^n}{n!}$$

Then divide by $(z-1)^3$.

(9) Use Residue theorem, answer is $2\pi i$.